On Generalized Hopf Differentials

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Abstract

A basic tool in the theory of constant mean curvature (cmc) surfaces in space forms is the holomorphic quadratic differential discovered by Heinz Hopf. However, for more general target spaces the $(2, 0)$- part of the second fundamental form of a cmc surface fails to be holomorphic.

The basic new result is that for cmc surfaces in the product spaces $S^2 \times R$ and $H^2 \times R$ holomorphicity can be restored with the help of explicit, geometrically defined correction terms.

Our generalized holomorphic quadratic differential is good enough to proceed along the lines of Hopf and prove that an immersed cmc sphere S^2 in such a product space must in fact be one of the embedded, rotationally-invariant surfaces described in the work of W.-Y. Hsiang and R. Pedrosa, which are the simplest cmc surfaces in the product spaces. The distance spheres do not have constant mean curvature any more.

The next step is to investigate the scope of the new construction. More precisely, we ask for which class of (oriented) Riemannian 3-manifolds (M^3, g) there exists a correction field L that induces a holomorphic quadratic differential on any immersed cmc surface. There is an amazingly simple necessary and sufficient condition, namely, L must satisfy a certain explicit inhomogeneous ODE system.

Integrability for this ODE system is by no means automatic; it rather imposes serious restrictions on the geometry of the 3-manifold, itself. A tedious classification reveals that solutions exist if and only if (M^3, g) is a homogeneous bundle with totally-geodesic fibers over some surface with constant curvature. Again an analogue to Hopf's theorem can be established.

The preceding results suggest that homogeneous 3-manifolds with at least 4-dimensional isometry groups are an appropriate setting for global results about minimal surfaces and cmc surfaces.

In order to test this thesis, we have started studying minimal surfaces in the Heisenberg group. If time permits, the talk will end with an outlook on these results.

Keywords: Minimal surfaces, surfaces with constant mean curvature, homogeneous bundles over surfaces, integrable distributions.