ENERGY AND TOPOLOGY OF SINGULAR UNIT VECTOR FIELDS ON $\mathbf{S}^3$

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The energy of a unit vector field $\vec{v}$ on a Riemannian manifold M is defined [4] as the energy of the section $X : M \to T^1M$. In terms of the Levi-Civita connection ∇, the energy of $\vec{v}$ is: $\mathcal{E}(\vec{v}) = \frac{n+1}{2}\mathrm{vol}(M) + \frac{1}{2}\int_M \|\nabla\vec{v}\|^2$. Note that $\mathcal{E}(\vec{v}) \geq \frac{n+1}{2}\mathrm{vol}(M)$ and only parallel fields attain the trivial minimum. The simplest spaces to be studied are, perhaps, odd-dimensional spheres.

We know [2] that the unit vector field on $\mathbf{S}^3$ tangent to the classical Hopf fibration is the unique field to minimize $\mathcal{E}$. In dimension 5 and higher, Hopf vector fields are unstable critical points of $\mathcal{E}$, see [5] and [3].

The infimum of $\mathcal{E}$ among all globally defined unit smooth vector fields on the spheres $\mathbf{S}^{2k+1}$, $k \geq 2$, is the energy of the radial vector field (that tangent to the geodesics from a fixed point), with two singularities, see [1].

In this work, we attempt to relate the energy and topology of vector fields with singularities.

Theorem. *Let $\vec{v}$ be a unit vector field on $\boldsymbol{S}^3$ with two antipodal singularities $\{N, S\}$. Then,*

$$\mathcal{E}(\vec{v}) \geq \Big(\big||\mathrm{Ind}_{\vec{v}}(N)| + |\mathrm{Ind}_{\vec{v}}(S)| - 1\big| + \frac{3}{2}\Big)\mathrm{vol}(\boldsymbol{S}^3),$$

where $\mathrm{Ind}_{\vec{v}}(P)$ denotes the index of $\vec{v}$ at the singularity $P \in \boldsymbol{S}^3$.

A similar result is not expected for higher dimensional spheres.

In a sense, the assumption of the Theorem about the quantity and the antipodal position of the singularities is a necessary restriction. It is possible to construct a sequence of unit vector fields, each with an arbitrary number of singularities in free positions, whose energy converges to the energy of the radial vector field.

References

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