On the Shape of of the Cubic Bézier Curve

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Abstract. A direct similarity of the Euclidean space $\mathbb{R}^n$, $n = 2, 3$ is an affine transformation $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which preserve the angles and the orientation of $\mathbb{R}^n$. The shape of a geometric object can be determined by the group of direct similarities. This means that two objects $F_1 \subset \mathbb{R}^n$ and $F_2 \subset \mathbb{R}^n$ have the same shape whenever there is a direct similarity $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $F_1 \overset{\varphi}{\mapsto} F_2$. In other words the shape of a geometric object can be considered as an equivalence class with respect to the group of direct similarities. There exist differential-geometric invariants which determine locally a plane or space curve up to a direct similarity (see [1], [2] and [3]). A triple differentiable plane curve with nonzero signed curvature is determined locally up to an orientation-preserving similarity by one differential geometric invariant. This invariant characterizes the shape of the plane curve. In the same way he shape of a space regular curve with nonzero curvature is determined locally by two differential invariants. On the other hand the cubic Bézier curves are widely used in computer graphics and computer aided geometric design (see [4] and [5]). Varying control points of Bézier curves and joining Bézier curves many authors found representations for shapes of different plane and space curves. In this work, we study the above invariants for cubic Bézier curves in the plane and the space. Thus we express explicitly the local shape of these curves by their control polygons. Using the matrix representation of a cubic Bézier curve we also discuss the change of its shape under an affine transformation.

Keywords: Differential geometry of curves, shapes of curves, Bézier curves.
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