An approach to classify submanifolds with parallel second fundamental form in riemannian symmetric spaces of arbitrary rank

Tillmann Jentsch

Let N be a riemannian symmetric space (simply connected, no euclidian factors). A parallel submanifold $M \subseteq N$ is by definition a submanifold, the second fundamental form h^M of which is parallel. The Codazzi equation implies that each tangent space T_pM is a curvature-invariant linear subspace of T_pN, and it follows by a theorem of Cartan that $M_0 := \exp_p(T_pM)$ is a totally geodesic submanifold of N with $T_pM_0 = T_pM$. Note that M_0 is parallel because $h^{M_0} \equiv 0$.

Thus we may ask the following question: Given some point $p \in N$ and some curvature invariant subspace $V \subseteq T_pN$, is there a connected parallel submanifold M with $p \in M$ and $T_pM = V$ which is not totally geodesic?

To give an answer to the above question let us introduce the holonomy Lie algebra of N

$$\mathfrak{k} := \mathrm{span}\{\, R(x,y) \,|\, x,y \in T_pN \,\} \subseteq \mathfrak{so}(T_pN)$$

the Lie subalgebra

$$\mathfrak{h} := \mathrm{span}\{\, R(x,y) \,|\, x,y \in V \,\} \subseteq \mathfrak{k} \ ,$$

which equips V and $V^\perp$ both with the structure of $\mathfrak{h}$-modules, and the following linear subspace

$$\mathfrak{k}_- := \{\, A \in \mathfrak{k} \,|\, \sigma_V \circ A = -A \circ \sigma_V \,\}) \ ,$$

where $\sigma \in \mathrm{O(T_pN)}$ denotes the linear reflection in V (i.e. $\sigma|V = Id_V$ and $\sigma|V^\perp = -Id_V^\perp$.)

Concerning extrinsic symmetric submanifolds in symmetric spaces (forming a subclass of all parallel submanifolds which is understood quite well), the following theorem is crucial for their classification:

Theorem 1 (Naitoh). *Suppose that V is strongly curvature invariant (i.e. V and $V^\perp$ both are curvature invariant subspaces of T_pN) . Then $M_0 := \exp_p(V)$ is extrinsic symmetric anyway. Moreover, if*

1. *V is an irreducible $\mathfrak{h}$-module and*

2. *there exists some extrinsic symmetric submanifold $M \subseteq N$ different from M_0 with $p \in M$ and $T_pM = V$,*

then

$$\dim(\mathfrak{k}_-) \geq \dim(V) \ . \tag{1}$$

Note that both sides of the above inequality do not explicitly depend on M ; given only the strongly curvature invariant space V, one can determine the dimension of $\mathfrak{k}_-$.

Example 1 (Naitoh)**.** Let $n \geq 2$ be some integer, $N = \mathbb{C}\mathrm{Q}_n := \{ [z_0, \ldots, z_{n+1}] \in \mathbb{C}\mathrm{P}_{n+1} \,|\, z_0^2 + \ldots + z_{n+1}^2 = 0 \}$ the n-dimensional complex quadric and $f : \mathrm{S}_n \to \mathrm{Q}_n,\ (x_0, \ldots, x_n) \mapsto [x_0, \ldots, x_n, i]$ the canonical two-fold covering from the real n-dimensional sphere. Then $M_0 := f(\mathrm{S}_n)$ is a totally geodesic submanifold of $\mathbb{C}\mathrm{Q}_n$, $V := f_* T_{(1,0,\ldots,0)}\mathrm{S}_n$ is a strongly curvature invariant subspace and an irreducible $\mathfrak{h}$-module. Naitoh calculates $\dim(\mathfrak{k}_-) = 1$, thus by the above theorem no extrinsic symmetric submanifold $M \subseteq N$ different from M_0 with $p := ([1, 0, \ldots, i]) \in M$ and $T_pM = V$ can exist.

However even if V is not strongly curvature invariant, I can prove the following theorem. But first notice: As $\mathfrak{h}$ is a reductive Lie algebra, we may write

$$\mathfrak{h} = Z_{\mathfrak{h}} \oplus [\mathfrak{h}, \mathfrak{h}] \ ,$$

where $Z_{\mathfrak{h}}$ denotes the center of $\mathfrak{h}$, and $[\mathfrak{h}, \mathfrak{h}] := \mathrm{span}\{ [A, B] \,|\, A, B \in \mathfrak{h} \}$ is a semisimple ideal. Furthermore, let us define:

$$\mathrm{Hom}_{[\mathfrak{h},\mathfrak{h}]}(V, V^{\perp}) := \{ \ell \in \mathrm{L}(V, V^{\perp}) \,|\, \forall\, A \in [\mathfrak{h}, \mathfrak{h}] : \ell \circ A|V = A \circ \ell \}$$

and $s := \dim(\mathrm{Hom}_{[\mathfrak{h},\mathfrak{h}]}(V, V^{\perp}))$; note that also s only depends on V .

Theorem 2. *Suppose that V and $V^{\perp}$ both are irreducible $\mathfrak{h}$-modules. Then $s \leq 4$ holds. If there exists some connected parallel submanifold $M \subseteq N$ with $p \in M$ and $T_pM = V$ which is not totally geodesic, then*

$$\dim(\mathfrak{k}_-) \geq \dim(V) - s \ . \tag{2}$$

If $s = 0$, then a parallel submanifold $M \subseteq N$ with $p \in M$ and $T_pM = V$ is a homogenous submanifold of N anyway .

I hope that one can use the above theorem in the same way as Naitoh's theorem to classify now the parallel submanifolds of N , provided one already knows the totally geodesic ones.

At the moment my work is in an experimental state; I don't even have examples at this point of time. Also I have no idea how large the dimension of the space $\mathfrak{k}_-$ can be, if V is not strongly curvature invariant.

As parallel submanifolds in space forms and also in the rank-1 symmetric spaces seem to be well understood, it would be interesting to apply our theory for rank-2 spaces, say the complex quadric.