

# HELICOIDAL SURFACES IN THE HEISENBERG 3-SPACE

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## 1. ABSTRACT

An interesting problem in geometry is to find surfaces with prescribed Gauss or mean curvature and which are obtained by various motions of curves around an axis which does not intersect the curve (e.g. rotation or translation). The problem has been studied for surfaces in the Euclidean and pseudo-Euclidean 3-space, in the Heisenberg 3-space. More precisely, in [Ca-Pi-Ra] R. Caddeo etc. studied surfaces of revolution with prescribed Gauss curvature in the 3-dimensional Heisenberg space. In the work [Ba-Kou] Ch. Baikoussis and Th. Koufogiorgos found explicitly the equations of helicoidal surface with prescribed Gauss or mean curvature in the 3-dimensional Euclidean space. Also in [Be-Kai-Pa1] Chr. Beneki etc. obtained a classification of surfaces of revolution with constant Gauss curvature in the 3-dimensional Minkowski space and in [Be-Kai-Pa2] obtained classification of helicoidal surfaces with prescribed Gauss and mean curvature in the 3-dimensional Minkowski space. The aim of the present work is to study helicoidal surfaces in the 3-dimensional Heisenberg space with prescribed Gauss curvature. If  $\gamma(u) = (u, 0, g(u))$ ,  $g(u) > 0$  is a curve in the 3-dimensional Heisenberg space  $H^3 = (R^3, ds^2)$  which lies in on the  $xz$ -plane, then a helicoidal surface is obtained by a rotation and translation of  $\gamma$  around  $z$ -axis. The 3-dimensional Heisenberg space  $H^3$  is the Riemannian  $(R^3, ds^2)$  where

$$(1) \quad ds^2 = dx^2 + dy^2 + [dz^2 + \frac{1}{2}(ydx - xdy)]^2.$$

This metric  $ds^2$  is a special case of a well-known 2-parameter family of metric introduced in 1897 by L. Bianchi given by

$$ds_{\lambda, \mu}^2 = \frac{dx^2 + dy^2}{1 + \mu(x^2 + y^2)^2} + (dz + \frac{\lambda}{2} \frac{ydx - xdy}{1 + \mu(x^2 + y^2)})^2, \lambda, \mu \in R.$$

For  $l = 1, \mu = 0$  it reduces to the metric (1). The parametric expression of such a surface is  $\vec{r}(u, v) = (u \cos v, u \sin v, g(u) + hv)$  where  $0 \leq v \leq 2\pi$  and  $h$  is the step of the translation. The Gauss curvature  $K$  of such surfaces turns out to be a function of the parameter  $u$  only, i.e.  $K = K(u)$ . The problem reduces to find solutions of differential equations whose integrals are rational expressions. In most cases these integrals can be only expressed in terms of elliptic functions. We present graphs of such surfaces in special cases as  $K=0$ ,  $K=\text{constant}$  and  $K$  an example of a simple function.

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## REFERENCES

- [**Ba-Kou**] : Chr. Baikoussis–T. Koufogiorgos: *Helicoidal surfaces with prescribed mean or Gaussian curvature*, Journal of Geometry , 63 (1998) 25–29.
- [**Be-Kai-Pa1**]: Chr. Beneki–G. Kaimakamis–B.J. Papantoniou: *A classification of surfaces of revolution with constant Gauss curvature in a 3-dimensional Minkowski space*, Bull. Cal. Math. Soc., 90 (1998) 441–458.
- [**Be-Kai-Pa2**]: Chr. Beneki–G. Kaimakamis–B.J. Papantoniou: *Helicoidal surfaces in three-dimensional Minkowski space*, J. Math. Anal. Appl., 275 (2002) 586–614.
- [**Ca-Pi-Ra**] : R. Caddeo–P. Piu–An. Ratto : *Rotational surfaces in 3-dimensional Heisenberg group with constant Gaussian curvature*, U.M.I., (7)(1996) 10-B, 341–357.
- [**Ei**] : L.P. Eisenhart : *An introduction to differential geometry*, Princeton University Press, 1947.

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