Abstract of a lecture for the PADGE Conference

On homogeneous geodesics in affine homogeneous manifolds

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For Riemannian homogeneous spaces, this topic was broadly studied by Z. Dušek, C. Gordon, A. Kaplan, O. Kowalski, S. Nikčevič, C. Riehm, J. Szenthe and L. Vanhecke, Z. Vlášek, recently also by D. Alekseevsky and A. Arvanitoyeorgos. By a *homogeneous geodesic* we mean a geodesic which is an orbit of a 1-parameter group of isometries. A Riemannian homogeneous space is called a *g.o. space* if all its geodesic are homogeneous (i.e. if "all geodesics are orbits"). It is known that every naturally reductive space is a g.o. space but the converse is not true. The first counter-example was found by A. Kaplan in dimension six, and, since then, all remaining 6-dimensional counter-examples (called *proper* g.o. spaces) were found. Higher-dimensional proper g.o. spaces have been also studied. In the opposite direction, the following theorem was proved: Every homogeneous Riemannian space admits at least one homogeneous geodesics, and this result can not be improved in general.

More recently, pseudo-Riemannian homogenous g.o. spaces started to be studied, let us mention the works by G. Calvaruso, Z. Dušek, O. Kowalski, R. Marinosci and by the physicists P. Meesen, S. Philip and others. For any homogeneous pseudo-Riemannian space, there is a well-founded (but still open!) conjecture saying that at least one homogeneous geodesics must exist. More papers were published about pseudo-Riemannian g.o. spaces and, very recently, it was discovered that there exist pseudo-Riemannian spaces which are g.o. spaces "up to measure zero". In contrast to the Riemannian case, more unusual phenomena appear for the pseudo-Riemannian case.

In both theories, the calculations are based on the so-called "Geodesic Lemma". It is a purely algebraic formula deciding, for each tangent vector of the given manifold $(G/H, g)$, if the corresponding geodesic is an orbit or not. In the pseudo-Riemannian case, an arbitrary parameter k comes into this formula, and it is needed if a tangent vector is a null-vector.

The Geodesic lemma is, in fact, equivalent to the following, non-algebraic condition: A Killing vector field Z on a space $(G/H, g)$ produces, as integral curves, homogeneous geodesics, if and only if $\nabla_Z Z = kZ$. (Here k is a constant which can be nonzero only if Z consists of null-vectors).

The last observation gives a direct way how to study homogeneous geodesics in the **affine case**. (According to the authors' knowledge, there were no serious attempts before to develop such a theory.)

We start with some basic concepts and results.

Proposition 1. Let M = G/H be a locally homogenous space with a left-invariant affine connection ∇. Then each orbit of a local 1-parameter subgroup $g_t \subset G$ on M is an integral curve of an affine Killing vector field on M.

Definition 1. A vector field Z on $M = (G/H, \nabla)$ is said to be *geodesic* if all its integral curves are geodesics up to a possible re-parametrization.

Proposition 2. A vector field Z on $M = (G/H, \nabla)$ is geodesic if, and only if, $\nabla_Z Z = fZ$, where f is a smooth function on M, still depending on the fixed vector field Z.

Putting together Proposition 1 and 2, we obtain

Proposition 3. If, for a Killing vector field Z, the formula $\nabla_Z Z = fZ$ is satisfied, then all integral curves of Z are homogeneous geodesics. The converse also holds.

The next general concepts and results are the following:

Definition 2. In a locally homogeneous affine manifold (M, ∇), by a *homogeneous geodesic* we mean a geodesic which is an orbit of an one-parameter local group of affine diffeomorphisms. (Here the canonical group parameter need not be the affine parameter of the geodesic). A *local affine g.o. space* is a locally homogeneous affine manifold (M, ∇) such that each geodesic is homogeneous.

Proposition 4. (M, ∇) is a local g.o. space if and only if it admits $n = \dim M$ independent geodesic Killing vector fields.

Then we started more detailed research in dimension 2, because we know the *explicit* classification of all locally affinely homogeneous connections in the plane domains. (This classification was a rather hard work. See B. Opozda for the torsion-less connections and T. Arias-Marco + O. Kowalski for connections with arbitrary torsion.)

We found conditions (in terms of the Christoffel symbols) saying that a given connection is an affine g.o.space. We also found an infinite family of locally affinely homogeneous connections which DO NOT admit any homogeneous geodesic. Thus, the problem which is open in the pseudo-Riemannian case is now definitely solved in the affine case. The details will be presented during the lecture.