Lie groupoid methods in classical field theory

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The importance of *Lie groupoids* for the study of discrete mechanics was first realised by Alan Weinstein [5] in 1996. Briefly speaking, Lie groupoids provide a unifying framework for many apparently different discrete mechanical systems, including systems with symmetry, nonholonomic constraints, or both (see [2]). In addition, many discrete integrable mechanical systems have a configuration space which is naturally a Lie groupoid. The latter include the pioneering efforts of Moser and Veselov [4], which provided the initial impetus for the work of Weinstein.

In this work, we use similar Lie groupoid methods to construct a geometric framework for the discretisation of classical field theories. The aim of this undertaking is twofold: to provide a sound geometric framework for a number of well-known results in discrete field theory [1, 3], as well as to uncover some new classes of discrete field theories (in particular, field theories with symmetry).

In our approach, discrete fields are modelled as Lie groupoid morphisms taking values in a certain "target groupoid" G, the specification of which is problem-dependent. A central element in our treatment is the set $\mathbb{G}^k$ of "k-gons" in G, whose elements, roughly speaking, can be viewed as k-gons whose sides are labelled with elements of G. Even though $\mathbb{G}^k$ is not a Lie groupoid (as it is not equipped with a partial multiplication), the remaining properties of $\mathbb{G}^k$ still allow us to construct a geometric theory of Lie groupoid field theories.

In particular, one can construct a *prolongation algebroid* $P^k\mathbb{G}$ over $\mathbb{G}^k$. Associated to a given Lagrangian L, there exist certain distinguished sections of the dual of this bundle, called *Poincaré-Cartan sections* which play a crucial role in the dynamics of discrete field theories. We derive the field equations for such a field theory, and show that their solutions are *multisymplectic* in a suitable sense. This is the field-theoretic analogue of symplecticity in discrete mechanics, and seems to be important for the construction of geometric numerical integrators (see [1, 3]).

Finally, we discuss aspects of symmetry in this framework and construct a reduction procedure. In the case of discrete field theories taking values in a Lie group $\mathcal{G}$ (a particular instance of a Lie groupoid) coinciding with the symmetry group, the dynamics can be interpreted in terms of aspects of discrete differential geometry, such as discrete $\mathcal{G}$-connections.

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[*] Joint work with F. Cantrijn.

References

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