The Mean Curvature of the Second Fundamental form

by
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If the second fundamental form of a surface in a three-dimensional semi-Riemannian space is positive-definite, this form can be studied as an abstract Riemannian metric. This requirement is equivalent with strict convexity for a surface in Euclidean space.

In this lecture I would like to present a study of the critical points of the area functional associated to the second fundamental form. These critical points are characterised by the vanishing of a scalar function, which will be called *the mean curvature of the second fundamental form.*

This curvature was introduced thirty years ago by E. Glässner and U. Simon for surfaces in Euclidean space. It can easily be seen that the area of the second fundamental form of such a surface can be computed as the integral of the square root of the Gaussian curvature, $\int \sqrt{K}\, d\Omega$.

On the other hand, the Gaussian curvature of the second fundamental form was already studied by E. Cartan. A theorem of R. Schneider characterises the spheres as the only closed, convex surfaces in $\mathbb{E}^3$ of which this Gaussian curvature of the second fundamental form is constant. This result can be seen as an extension of a famous theorem due to H. Liebmann. Several other characteristic properties of spheres, involving the mean and Gaussian curvature of the second fundamental form, as well as the classical mean and Gaussian curvature, were found by C. Baikoussis, T. Hasanis, T. Koufogiorgos, U. Simon, G. Stamou and others. Some of these results have been generalised to surfaces in semi-Riemannian manifolds by J.A. Aledo, S. Haesen, L.J. Alias, A. Romero and others.

In the present study, several extensions of the above results involving the mean curvature of the second fundamental form, have been obtained for surfaces in semi-Riemannian manifolds. Furthermore, an integral inequality has been proved, in which the Ricci tensor of the ambient space and the Euler-Poincaré characteristic of the surface are involved. Equality is achieved if and only if the surface is totally umbilical.

Finally, I would like to mention that an integral equality of J.H. Jellett and H. Minkowski, which involves the area of a convex, compact surface in Euclidean space, has been adapted to the area of the second fundamental form. This equality permitted a further characterisation of the spheres in Euclidean space to be proved.